Basic Equations of Fluid Flow

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- Mass Balance

- Momentum Balance

- Mechanical Energy Balance
Basic Concepts

- Conserved Quantities
  - Chemical species
  - Mass
  - Momentum
  - Energy

- Law of Conservation of Quantities
  - Conservation of Chemical species
  - Conservation of Mass
  - Conservation of Momentum
  - Conservation of Energy
Basic Concepts

- **Rate Equation**
  - It describes the transformation of conserved quantity.
  - Transformation of conserved quantity is based on specified unit of time (Rate).

- **Components of Rate Equation**
  - Input
  - Output
  - Generation
  - Consumption
  - Accumulation
Basic Concepts - Characteristics

- Independent of the level of application
- Independent of the coordinate system to which they are applied
- Independent of the substance to which they are applied
Basic Concepts - Application

- Balances
  - Control Volume
  - Control surface

- Types of Balances
  - Overall Balance
  - Differential Balance
The notation of conserved quantity is $\varphi$

$$\varphi = \varphi(t, x, y, z)$$

$x, y \& z =$ three independent space variables

t = one independent time variable
Basic Concepts - Definition

✓ Steady-state
✓ Uniform
✓ Equilibrium
✓ Flux
1. Inlet and Outlet terms
2. Generation and consumption term
3. Accumulation term
Case I: Steady state transport without regeneration

Case II: Steady state transport with regeneration
Equation of Continuity

Volume element $\Delta x \Delta y \Delta z$

Apply Law Of Conservation Of Mass On This Small Volume Element
**Streamline** – An imaginary curve in a mass of flowing fluids where at every point on the curve the net-velocity vector is tangent.

- No net flow across streamline

**Stream tube** – tube of small and large cross section that is entirely bounded by streamlines

- Like imaginary pipe in flowing fluid
- No net flow across the surface
A Stream tube, or stream filament, is a tube of small or large cross section and of any convenient cross-sectional shape that is entirely bounded by streamlines.

A stream tube can be visualized as an imaginary pipe in the mass of flowing fluid.

\[ \dot{m} = \rho_a u_a S_a = \rho_b u_b S_b \]

\[ \dot{m} = \rho u S = \text{const} \]
The average velocity of the entire stream flow through cross-sectional area $S$:

$$\bar{V} \equiv \frac{m}{\rho S} = \frac{1}{S} \int_S u \, dS$$

$$\bar{V} = \frac{q}{S}$$

Case: flow through circular cross-section

$$\frac{\rho_a \bar{V}_a}{\rho_b \bar{V}_b} = \left(\frac{D_b}{D_a}\right)^2$$
Mass Velocity

\[ \bar{V} \rho = \frac{\dot{m}}{S} \equiv G \]

- G is independent of temperature and pressure when the flow is steady and the cross section is unchanged.
- Significant for compressible fluids.
- Mass velocity – Mass current density or Mass flux
- Average velocity – Volume flux
Example 4.1. Crude oil, specific gravity 60°F/60°F = 0.887, flows through the piping shown in Fig. 4.2. Pipe A is 2-in. (50-mm) Schedule 40, pipe B is 3-in. (75-mm) Schedule 40, and each of pipes C is 1\(\frac{1}{2}\)-in. (38-mm) Schedule 40. An equal quantity of liquid flows through each of the pipes C. The flow through pipe A is 30 gal/min (6.65 m³/h). Calculate (a) the mass flow rate in each pipe, (b) the average linear velocity in each pipe, and (c) the mass velocity in each pipe.
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Example 4.2

- Air at 20 C and 2 atm absolute pressure enters a finned-tube steam heater through a 50-mm tube at an average velocity of 15 m/s. It leaves the heater through a 65-mm tube at 90 C and 1.6 atm absolute. What is the average air velocity at the outlet?
"The sum of forces acting in the $x$ direction equals the difference between the momentum leaving with the fluid per unit time and that brought in per unit time by the fluid."

$$\sum F = \frac{1}{g_c} (\dot{M}_b - \dot{M}_a)$$
Total momentum flow is not equal to what calculated by product of mass flow rate and average velocity.

Correction factor is introduced.

From convective momentum flux, for differential cross-section $dS$

$$\frac{dM}{dS} = (\rho u)u = \rho u^2$$

For whole stream

$$\frac{\dot{M}}{S} = \frac{\rho \int_{S} u^2 dS}{S}$$
Momentum correction factor is defined as

\[ \beta = \frac{\dot{M}/S}{\rho \overline{V^2}} \]

By substitution

\[ \beta = \frac{1}{S} \int_S \left( \frac{u}{\overline{V}} \right)^2 dS \]
Momentum of Total Stream

\[ \sum F = \frac{\dot{m}}{g_c} (\beta_b \bar{V}_b - \beta_a \bar{V}_a) \]

**Note:** All forces components acting on the fluid is in the direction of velocity component.

**Forces:**

- Pressure forces
- Shear stress at the boundary
- Gravitational force

\[ \sum F = p_a S_a - p_b S_b + F_w - F_g \]
Momentum Balance in Potential Flow: The Bernoulli Equation without Friction

- Steady flow
- Potential flow
- Frictional effects are not considered
- Increasing cross-section
- Direction of flow from $a$ to $b$
- Constant mass flow rate $\dot{m}$
\[ \sum F = \frac{\dot{m} \Delta u}{g_c} \]

\[ p_a S_a = p S \]

\[ F_g = \frac{g}{g_c} \overline{S} \rho \Delta L \cos \phi \]

\[ \sum F = p_a S_a - p_b S_b + F_w - F_g \]

\[ p_b S_b = (p + \Delta p)(S + \Delta S) \]

\[ F_w = \int_0^{\Delta S} \dot{p} \, dS = \overline{\dot{p}} \Delta S \]
\[ \cos \phi = \frac{\Delta Z}{\Delta L} \]

By substituting

\[ \dot{m} \frac{\Delta u}{g_c} = \Delta S (\bar{p}' - p) - S \Delta p - \Delta p \Delta S - \frac{g}{g_c} \bar{S} \rho \Delta Z \]

Dividing by \( \rho S \Delta L \)

\[ \frac{\dot{m}}{g_c \rho S} \frac{\Delta u}{\Delta L} = \frac{\bar{p}' - p}{\rho S} \frac{\Delta S}{\Delta L} - \frac{1}{\rho} \frac{\Delta p}{\Delta L} + \frac{\Delta S}{\rho S} \frac{\Delta p}{\Delta L} - \frac{g}{g_c S} \frac{\bar{S}}{\Delta L} \frac{\Delta Z}{\Delta L} \]
Applying limits

\[
\frac{\dot{m}}{g_c \rho S} \frac{du}{dL} = - \frac{1}{\rho} \frac{dp}{dL} - \frac{g}{g_c} \frac{dZ}{dL}
\]

By substituting

\[
\dot{m} = u \rho S
\]

\[
\frac{u \rho S}{g_c \rho S} \frac{du}{dL} = - \frac{1}{\rho} \frac{dp}{dL} - \frac{g}{g_c} \frac{dZ}{dL} = 0
\]

\[
\frac{1}{\rho} \frac{dp}{dL} + \frac{g}{g_c} \frac{dZ}{dL} + \frac{d(u^2/2)}{g_c dL} = 0
\]
\[
\frac{dp}{\rho} + \frac{g}{g_c} dZ + \frac{1}{g_c} d\left(\frac{u^2}{2}\right) = 0
\]

\[
\frac{p_a}{\rho} + \frac{g Z_a}{g_c} + \frac{u_a^2}{2g_c} = \frac{p_b}{\rho} + \frac{g Z_b}{g_c} + \frac{u_b^2}{2g_c}
\]
- Bernoulli equation - special form of mechanical energy balance
- All the terms in this equation are scalar and have the dimensions of energy per unit mass
- Mechanical potential energy
- Mechanical kinetic energy
- Mechanical work done
Example 4.2. Brine, specific gravity $60^\circ F/60^\circ F = 1.15$, is draining from the bottom of a large open tank through a 50-mm pipe. The drainpipe ends at a point 5 m below the surface of the brine in the tank. Considering a streamline starting at the surface of the brine in the tank and passing through the center of the drain line to the point of discharge and assuming that friction along the streamline is negligible, calculate the velocity of flow along the streamline at the point of discharge from the pipe.
Bernoulli Equation: Correction for Effects of Solid Boundaries

- Correction of the *kinetic-energy term* for the variation of local velocity $u$ with position in the boundary layer.

- Correction of the equation for the existence of *fluid friction*, which appears whenever a boundary layer forms.
Kinetic Energy of Stream

\[ d\dot{E}_k = (\rho u \, dS) \frac{u^2}{2g_c} = \frac{\rho u^3 \, dS}{2g_c} \]

\[ \dot{E}_k = \frac{\rho}{2g_c} \int_S u^3 \, dS \]

\[ \frac{\dot{E}_k}{\dot{m}} = \frac{(1/2g_c) \int_S u^3 \, dS}{\int_S u \, dS} = \frac{(1/2g_c) \int_S u^3 \, dS}{V S} \]
Kinetic Energy Correction Factor

\[ \frac{\alpha \bar{V}^2}{2g_c} = \frac{\dot{E}_k}{\dot{m}} = \int_S u^3 \, dS / 2g_c \bar{V} S \]

\[ \alpha = \frac{\int_S u^3 \, dS}{\bar{V}^3 S} \]

If \( \alpha \) is known, the average velocity can be used to calculate the kinetic energy from the average velocity by using \( \alpha \bar{V}^2 / 2g_c \) in place of \( u^2 / 2g_c \). To calculate the

\[ \frac{p_a}{\rho} + \frac{g Z_a}{g_c} + \frac{\alpha_a \bar{V}_a^2}{2g_c} = \frac{p_b}{\rho} + \frac{g Z_b}{g_c} + \frac{\alpha_b \bar{V}_b^2}{2g_c} \]
Fluid friction can be defined as any conversion of mechanical energy into heat in a flowing stream.

In frictional flow the quantity \( \frac{p}{\rho} + \frac{u^2}{2g_c} + \frac{g}{g_c} Z \) is not constant along a streamline but always decreases in the direction of flow.
Correction of Bernoulli Equation for Fluid Friction

- The term $h_f$ represents all the friction generated per unit mass of fluid.
- The unit of $h_f$ is energy per unit mass.
- Different from other terms in two ways:
  - Not at specific location but at all points.
  - Not inter-convertable.
- $h_f$ includes both skin friction and form friction.
Skin Friction

- Friction generated in unseparated boundary layers is called skin friction.

- Friction appears in boundary layers because the work done by shear forces in maintaining the velocity gradients in both laminar and turbulent flow is eventually converted into heat by viscous action.
Form Friction

- When boundary layers separate and form wakes, additional energy dissipation appears within the wake, the friction of this type is called form friction.

- Form friction is a function of the position and shape of the solid.
Example 4.3. Water with a density of 998 kg/m³ (62.3 lb/ft³) enters a 50-mm (1.969-in.) pipe fitting horizontally, as shown in Fig. 4.5, at a steady velocity of 1.0 m/s (3.28 ft/s) and a gauge pressure of 100 kN/m² (2088.5 lb/ft²). It leaves the fitting horizontally, at the same elevation, at an angle of 45° with the entrance direction. The diameter at the outlet is 20 mm (0.787 in.). Assuming the fluid density is constant, the kinetic-energy and momentum correction factors at both entrance and exit are unity, and the friction loss in the fitting is negligible, calculate (a) the gauge pressure at the exit of the fitting and (b) the forces in the x and y directions exerted by the fitting on the fluid.
A pump is used in a flow system to increase the mechanical energy of the flowing fluid, the increase being used to:

- maintain flow,
- provide kinetic energy,
- offset friction losses and
- sometimes increase the potential energy

\[ W_p - h_{fp} \equiv \eta W_p \]

\[ \eta = \frac{W_p - h_{fp}}{W_p} \]
Pump Work in Bernoulli Equation

\[
\frac{p_a}{\rho} + \frac{gZ_a}{g_c} + \frac{\alpha_a V_a^2}{2g_c} + \eta W_p = \frac{p_b}{\rho} + \frac{gZ_b}{g_c} + \frac{\alpha_b V_b^2}{2g_c} + h_f
\]
Example 4.4. In the equipment shown in Fig 4.6, a pump draws a solution of specific gravity 1.84 from a storage tank through a 3-in. (75-mm) Schedule 40 steel pipe. The efficiency of the pump is 60 percent. The velocity in the suction line is 3 ft/s (0.914 m/s). The pump discharges through a 2-in. (50-mm) Schedule 40 pipe to an overhead tank. The end of the discharge pipe is 50 ft (15.2 m) above the level of the solution in the feed tank. Friction losses in the entire piping system are 10 ft-lbf/lb (29.9 J/kg). What pressure must the pump develop? What is the power of the pump?